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On Graphical Methods of determining the Local or Greenwich Time of Sunset at different places within a given region.

By H. H. Turner, D.Sc., F.R.S., Savilian Professor.

1. A few years ago my attention was directed in two very different ways to the problem of determining by some simple method—preferably a graphical one—the time of sunrise or sunset at various places within a limited region. First, Mr. Robert Sewell, in discussing Indian eclipses, wanted a correction for reducing the time of sunset or sunrise at any place in India to the time of sunset at Lanka. Secondly, it was decided that cyclists were to be guided in lighting their lamps by the time of local sunset, and not by that of sunset at Greenwich. It was easy to devise *some* graphical method, but it was not so easy to arrive at one which seemed to be final, for new, though slight, simplifications were apt to suggest themselves. In 1902 August I had arrived at the form described below, and no modification of value has since occurred to me; but in 1903 September, during the visit of the British Association to Belfast, I found that Mr. D. E. Benson and Mr. G. Napier Clark, who had also been working at the problem for some years, had arrived by a different route at a somewhat similar solution, of which a brief description is appended.

2. The ordinary processes involve two corrections, one for longitude and one for latitude. The former is a mere addition or subtraction; the latter is made usually by spherical trigonometry, put into the form of tables perhaps. To combine the two, remark that the places where the Sun is setting at any instant lie on a great circle of the Earth, if we neglect refraction: or in a small circle parallel to it at a distance of about 33′ if we take in refraction. As the Earth rotates, this circle travels over its surface, maintaining constant its position relatively to successive meridians.

3. Thus if we take any map on which the meridians are represented by parallel lines at equal distances for equal differences of longitude, and draw the sunset-line for any moment, its position for any other moment is obtained by simply sliding the sunset-line uniformly across the map in a direction perpendicular to the meridians. Thus, in fig. 1, if AB be the sunset-line on any date at Greenwich sunset, then by sliding it a horizontal distance *a*A, so as to pass through Hull, we obtain the sunset-line at Hull sunset, and the distance A*a* measures the elapsed time.

4. But the sunset-line will on most maps be a curved line, excepting at an equinox, when it coincides with a meridian, if refraction be neglected. (If refraction be included there will generally be some date near an equinox for which the sunset-line is straight.) We can choose the form of the map so that the

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sunset-line will be straight at some one particular date, for we can set our lines of latitude on any arbitrary scale, but at other dates the lines will still be curved. As this point is important we will examine it in detail before proceeding further.

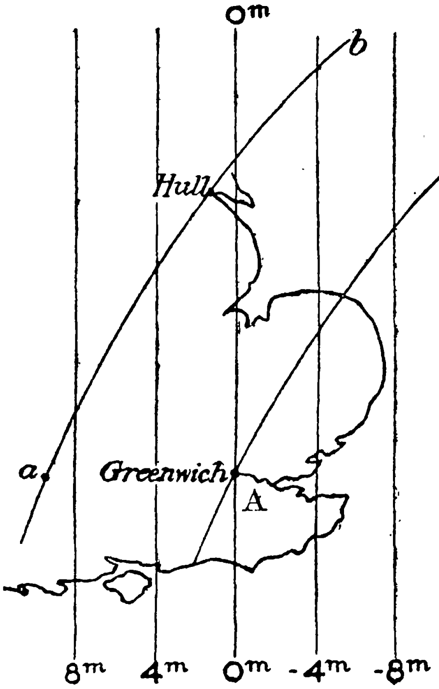

FIG. I.

5. Neglecting refraction, the hour-angle of the Sun at setting differs from 90° by an angle

$$\theta = \sin^{-1}(\tan\phi \tan\delta)$$

where δ is the Sun's declination, and ϕ the latitude of the place. The hour-angle represents local time. If l be the longitude of the place, the G.M.T. of sunset will be

$$l + \theta$$

where l and θ are both expressed in time.

Since l increases uniformly in one coordinate, θ must increase uniformly in the other if we are to have a straight line; and thus we must lay off our latitudes so that

$$\sin^{-1}(\tan\phi \tan\delta)$$

increases uniformly. Now it is easily seen that if this is true for one value of δ (not zero) it will not be true for another.

6. But when we are dealing with a limited area—such as Great Britain, say—we can make all the lines *sensibly* straight, since δ is never greater than $23\frac{1}{2}$°. In illustration of this point let us make the lines straight when $\delta = 26° 36'$ and see what the de-

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viation is when $\delta' = 14^\circ\ 5'$, which should give about the maximum error, remembering that $\delta = 0^\circ$ is always straight. These values of δ are chosen because $\tan 26^\circ\ 36' = \cdot 5$ and $\tan 14^\circ\ 5' = \cdot 25$, which makes the multiplications $\tan\phi \tan\delta$ easy. In Table I. column A gives the values of θ for $\delta = \tan^{-1} 0\cdot 5$, and column B the values of θ for $\delta = \tan^{-1} 0\cdot 25$. By comparing the two we find that, *approximately*,

$$B = 4^\circ\!\cdot 4 + \cdot 359\ A$$

and the values calculated from this formula are given in column C. The difference B—C never exceeds $0^\circ\!\cdot 2$ or $0^{\mathrm{m}}\!\cdot 8$, which may perhaps be neglected for our present purpose.

Table I.

ϕ	A.	B.	C.	B—C.
°	°			°
50	36·6	17·3	17·5	−0·2
52	39·8	18·7	18·7	0·0
54	43·5	20·1	20·0	+0·1
56	47·8	21·8	21·6	+0·2
58	53·2	23·6	23·5	+0·1
60	60·0	25·7	25·9	−0·2

Thus between parallels 50° and 60° we can use straight sunset-lines without any great error if we lay off our parallels of latitude so as to make the extreme case of $\delta = 27^\circ$ straight. The error will be much less if we make a map of England alone, which extends from 50° to 56° only.

7. We have taken limits 50° to 60° of latitude in the above example. In Table II. we see that the proposition is very nearly true from *the equator* up to 60° for values which δ can actually assume. Taking $\tan\delta_1 = \frac{1}{3}$ (or $\delta = 18\frac{1}{2}^\circ$ about) as the case for actual straightness, the greatest deviations from straightness will occur when $\tan\delta_2 = \frac{1}{6}$, say, and $\tan\delta_3 = \cdot 44$ ($\delta_3 = 23\frac{1}{2}^\circ$). Column D gives the value of θ for δ_1; column E for δ_2, which is compared with $\cdot 49\,D$; and column F for δ_3, which is compared with $1\cdot 35\,D$.

Table II.

ϕ.	D.	E.	E—·49 D.	F.	F—1·35 D.
°	°	°	°	°	°
0	0·0	0·0	0·0	0·0	0·0
10	3·4	1·7	0·0	4·4	−0·2
20	7·0	3·5	+0·1	9·2	−0·2
30	11·1	5·5	+0·1	14·7	−0·3
40	16·3	8·1	+0·1	21·6	−0·4
50	23·4	11·5	0·0	31·6	0·0
60	35·3	16·8	−0·5	49·6	+1·9

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The extreme error in the last case is $1°{\cdot}9$, or nearly 8^{m}, which can scarcely be neglected; but it is evident that it is only near 60° that the error becomes large.

8. But now returning to a special problem, such as a sunset diagram for England, we have established this proposition.

When meridians are represented on a map by equidistant parallel lines it is possible to choose a variable scale of latitude so that the sunset-lines at all dates are straight within limits which we may neglect.

The proposition has been examined, for simplicity's sake, when refraction is neglected; but since refraction only slightly modifies the lines, it is readily seen to be also true when refraction is taken into account.

9. To make the best latitude scale for the purpose we may proceed as follows:—

Select the date for which $\tan\delta = \frac{1}{3}$ ($\delta = 18\frac{1}{2}°$) as that for which to make the line accurately straight, or some date near this. We shall get the maximum deviations from straightness at about $\delta = 9°$ (since $\delta = 0°$ is also straight) and $\delta = 23\frac{1}{2}°$, as in Table II.

Calculate the accurate times of sunset, including refraction, for each degree of latitude on this date, and let them be as follows:—

Latitude	50°	51°	52°	53° . . .
Local time of sunset	t_0	t_1	t_2	t_3 . . .

Then the space between 50° and 51° is to be proportional to $t_1 - t_0$; that between 51° and 52° to $t_2 - t_1$; and so on. This enables us to set out our latitude-lines.

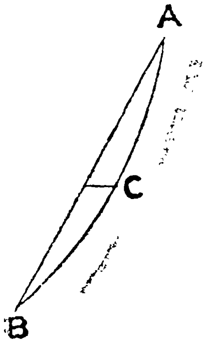

FIG. 2.

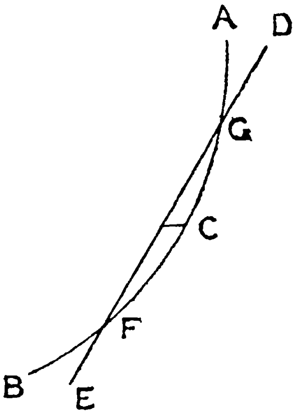

FIG. 3.

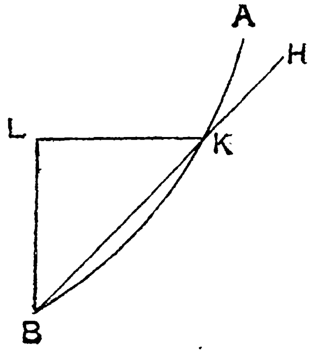

FIG. 4.

10. To find the sunset-line for any date.—We are really representing a curved line by a straight one, and can make the compromise in a variety of ways, none of them deviating much from the truth, for the straight line and the curve will cut at two points which may be varied in position. If we take these

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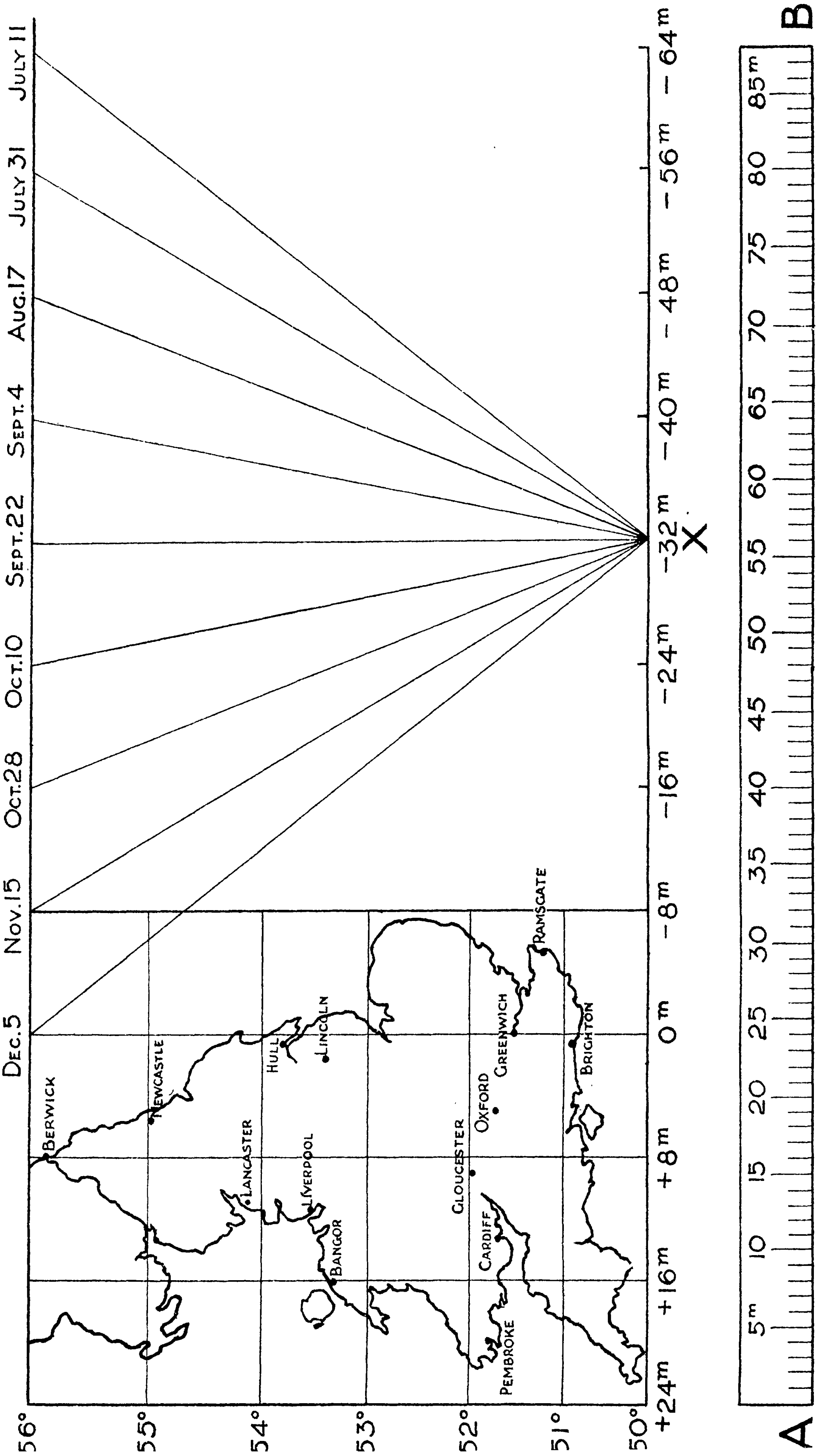

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points of intersection at the top and bottom of the map, as in fig. 2, the error will be greatest in the middle, and always of one sign. It will be better to put the points of intersection, as in fig. 3, so that DG = EF = $\frac{1}{6}$ DE, and then we shall distribute the error. But if we are drawing a series of lines from a fixed point B the best compromise may be BKH, as in fig. 4. The general principle of getting a good compensation will be obvious from these illustrations. In practice the rule comes to this: Select two definite latitudes, either those of F and G or those of B and K, as the case may be, and find the difference in time of sunset for these two latitudes at any date. If BL be a meridian, set off LK equal to this difference on the latitude-line through L; then BK will be the sunset-line for the date.

11. We have now considered everything but the *size* of the map, which is an important practical detail, and determines the scale of time. We may represent the G.M.T. of sunset in two ways:

(*a*) Either the map shall show the actual G.M.T. of sunset, or

(*b*) It shall show the *difference* which is to be added to (or subtracted from) the time of sunset at Greenwich, which can be given in a separate table.

12. If we adopt (*a*), then, since the time of sunset at Greenwich itself varies more than four hours, and that at Berwick more than five hours, with a map 6 inches long we can only allow 1 inch to an hour. If we adopt (*b*), the whole *difference* to be represented is only about 80 minutes, so that we can have nearly 4 inches to the hour. It is true that method (*b*) involves adding two quantities together instead of taking a single quantity from the map; but it seemed to me that the advantages of an open scale were too great to be neglected.

13. Having settled the scale of longitude, that of latitude follows generally from the consideration that the extreme sunset-lines should be inclined about 45° to the meridians—not more, and preferably less.

14. I will now give an indication of the form which I had arrived at for the map last year (see Plate 12).

Method (*b*) being adopted, it was determined to represent the difference of sunset-time *not* from that at Greenwich, which involves + and — signs, but from a point X, 32^{m} east of Greenwich, and in latitude 50°, so that the correction to sunset at X should always be additive.

The sunset-lines are represented by straight lines through X, the times of sunset being accurately calculated for the parallel 54°.

The scale of latitude expands as we go north, being accurately proportional to the differences of sunset-time for $\tan \delta = \frac{1}{3}$.

A certain number of the lines are drawn corresponding to dates shown at the top; those for intermediate dates can be

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inferred. The other half of the year is shown on another scale of dates, which can be folded down over that represented in the figure.

Supplementing the map there must be given a table of times of sunset for the point X.

To find the time of sunset at any point P for any date, take the paper scale AB (which is a scale of minutes and should be detached from the map), lay the corner A on P, keeping the scale horizontal, and read where it cuts the date-line. This reading gives the number of minutes to be added to the time of sunset at X, which is given in the table.

15. The scheme of Mr. Benson and Mr. Clark may be indicated as follows :—

(*a*) They take Mercator's projection for the map. The latitude scale is a gradually expanding one, and happens therefore to be nearly the same as that we were led to theoretically in what precedes. Mr. Benson and Mr. Clark determined its suitability for straight sunset-lines by experiment.

(*b*) They place the reference point X in the centre of the map, which involves the use of + and − signs, but reduces the size of the map.

(*c*) The sunset-lines are not actually drawn. The dates are indicated at the top and bottom of the map, and a string fastened at X can be stretched to the date so as to form any required sunset-line.

(*d*) No separate scale AB is given. The distance of a place to right or left of the line is to be inferred from the distances between meridians. This is indeed not difficult when the point X is placed in the middle of the map, and the distances are all small.

(*e*) The table giving sunset at X is practically ranged alongside the dates given at the top and bottom of the map, in the form of an accompanying time-scale. This is in many ways convenient, and the only disadvantage is (as indicated in § 11) that the scale is rather small, as four hours are to be represented within the dimensions of the map.

Tables to facilitate the working of Combined Altitudes by Saint-Hilaire's Method. By Lieut. E. B. Simpson-Baikie, R.N.R.

Communicated by the Secretaries.

The most valuable of all methods for finding a ship's position at sea is no doubt the combined altitude method, which consists of either a combination of simultaneous observations of different heavenly bodies, or a combination of successive observations of the same one.